

A Generalized Golden Rule Representative Value for Multiple-Criteria Decision Analysis

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Abstract—Multicriteria decision analysis evaluates multiple conflicting criteria in decision making, but conflicting criteria are typical in evaluating options. As the existing ordering operations involved in multicriteria decision making cannot easily be implemented with intervals, we assume that scalar representative values with intervals can effectively avoid this issue. To deal with interval-valued criteria, we propose a generalized golden rule representative value approach, which involves the sigmoid function of backpropagation neural networks to tune parameters. Our approach considers the uncertainties and side effects of the interval variables to improve individual scalar representative values. Based on numerical examples, we address the effectiveness of the proposed approach, and we provide a specific application concerning multicriteria decision making with interval criteria satisfaction.

Index Terms—Comparison system, golden rule representative value, interval valued, multicriteria decision function, sigmoid function, uncertainty.

I. INTRODUCTION

MCDM has been paid close attention in information systems in recent years [1]–[3]. The applications of MCDM have been widely discussed in many fields, such as supplier selection [4], [5], fault diagnosis [6], [7], and heart-care management [8], [9]. To make the decision outcome reliable, some techniques are applied to its optimization processes, including analytical hierarchy process (AHP) [10]–[12], decision making trial and evaluation laboratory (DEMATEL) method [13], [14], and technique for order preference by similarity to an ideal solution (TOPSIS) method [15]–[17]. The estimation outcome is assessed by combining the experts' synthesized opinions, it is vital to consider the uncertain

information in the learning process. In this case, existing studies addressed effective techniques to handle the uncertainty better, such as Dempster–Shafer evidence theory (DSET) [18]–[20], information entropy theory [21], [22], intuitionistic fuzzy set (IFS) theory [23]–[25], and hesitant fuzzy set (HFS) theory [26]. A multicriteria decision function can be used to aggregate the satisfaction of each alternative with respect to the individual criteria to obtain the alternative overall satisfaction to a decision problem [27], [28]. In addition, the interval value allows to represent uncertainty information, which attaches great importance to decision making in a dynamic environment [29], [30]. For instance, the interval-valued Pythagorean fuzzy set (IVPFS) [31] or interval-valued HFS (IVHFS) [17] can be considered as an effective extended model. Thus, if the individual criteria satisfaction by all alternatives is provided in terms of an interval value, reasonably obtaining the ordering is regarded as a very essential issue.

Traditionally, the method to measure the size of the interval simply uses the midpoint m . However, in some special situations, this method will lead to some misunderstandings. Although the interval range is very large, as long as the interval median points are the same, the traditional *midpoint* method results indicate that their scalar is the same. Therefore, the *midpoint* method is not considered appropriately for the interval ranges. Many scholars have discussed this problem [32]–[35]. Compared with other methods, the intervals are generally converted into representative values and then ordered by comparing their scalar values.

The interval midpoint and its range are the two most important eigenvalues for any interval; thus, the representative value of an interval mainly focuses on these two factors. Yager [36], [37] proposed a rule-based golden rule approach to obtain the representative value of the interval that can implement the required ordering and comparing operations when interval-valued information is given. Moreover, it can be seen that the result of the Sugeno integral [38], [39] is a special case of this model. However, while the representative values obtained according to this method can reflect some important features of the interval, they ignore the relative influence between the intervals in the comparison system and the uncertainty caused by the interval representation.

Considering the above problems, we introduce the sigmoid function [40] and the Takagi, Sugeno, and Kang (TSK) fuzzy model [41], which is commonly applied in biology and information science, to build an improved model for dealing with such situations. Due to its special characters, the sigmoid

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function has been focused on evolving fields, such as machine learning and neural networks. The TSK fuzzy model is one of the most famous fuzzy systems with a good nonlinear approximation ability and is widely used in many applications, such as system identification, pattern recognition, and image processing. In this paper, a new defined formula based on the sigmoid function is proposed to better express the correction relationship of the impact on the range. By introducing some new parameters to represent the inherent uncertain information brought by the range of the interval, the proposed method can be used to dynamically compare different intervals with more reasonable effects.

The rest of this paper is organized as follows. In Section II, some basic concepts about the TSK fuzzy model, the sigmoid function, and the golden rule representative value are introduced. In Section III, a new method for ordering the intervals based on the rule-based approach is proposed. In Section IV, some numerical examples are shown to discuss several different cases and to show the effectiveness of the approach. In Section V, some applications with interval-valued criteria satisfaction for the alternatives are shown to illustrate the practicality in multicriteria decision-making problems. In Section VI, the comparisons and discussion are given. And in Section VII, the sensitivity analysis is utilized to illustrate the robustness and sensitiveness. Finally, in Section VIII, some conclusions of the proposed method are drawn.

II. PRELIMINARIES

In this section, some basic concepts which include the golden rule, the TSK fuzzy model, and the sigmoid function are introduced.

A. TSK Fuzzy Model

The TSK fuzzy system, first proposed by Takagi and Sugeno [41], has a good nonlinear approximation ability and is widely used in many fields, such as system identification, classification [42], [43], and image processing [44]–[46]. In addition, some bionic algorithms are also widely used in fuzzy system modeling [47]. However, the TSK fuzzy model has always been regarded as one of the most famous fuzzy systems defined by the *IF-Then* rule. Some of the basic concepts are as follows.

Definition 1: The basic form of the TSK fuzzy model is shown as follows [41]:

R_i : If x_1 is A_{i1} , x_2 is A_{i2} and \dots x_j is A_{ij} and \dots x_n is A_{in}
Then, $y_i = f_i(X)$

where $X = x_1, x_2, \dots, x_n$ are the predecessor variables and A_{ij} is the fuzzy set of x_j in Rule i . $f_i(X)$ represents the post-component variables. The specific expression is shown as follows:

$$f_i(x) = a_0^i + a_1^i x_1 + \dots + a_n^i x_n \quad (1)$$

where R numerically expresses the total number of rules.

Definition 2: When the input of X satisfies the condition that $X = [x_1, x_2, \dots, x_n]$, the product method is introduced for T norms to the predecessor variables. In addition, we use the

center-of-gravity method to defuzzify. According to the above, the output of the TSK fuzzy model is expressed as [41]

$$y = \frac{\sum_{i=1}^R \prod_{j=1}^n A_{ij}(x_j) y_i}{\sum_{i=1}^R \prod_{j=1}^n A_{ij}(x_j)}. \quad (2)$$

Since the TSK fuzzy model was proposed, it has been shown to have broad application prospects in the field of state estimation due to its characteristics.

The conventional forecasting method has difficulty in meeting the actual requirements of the site for the following reasons.

- 1) The changes caused by the situation in the future cannot be completely grasped in advance.
- 2) Even though we know that certain complex factors have an impact on the system, it is still difficult to obtain quantitatively and to accurately determine its impact.
- 3) Many practical problems depend on the judgment ability and experience of the forecasters. The traditional methods without artificial intelligence technology require manual intervention to meet the actual forecasting requirements.

Compared with the traditional method, the TSK fuzzy model is based on rules. We can use the data information such as error and expert knowledge as the basic information, which provides flexibility for designing appropriate correction subsystems in prediction systems. Thus, the TSK fuzzy model is widely used in many applications.

B. Golden Rule Representative Value

To model complexity in the real world, many methods, such as complex networks [48], [49], nature-inspired methods [50]–[53], and neural networks [54], [55] have been presented. Additionally, using some existing rules, such as the golden rule is an efficient method for modeling complexity in the real world. In this section, the golden rule model presented by Yager is briefly introduced. The intervals often play an important role in many situations, especially in MCDM. In anticipation of solving the MCDM problems in situations in which the satisfaction of individual criteria by an alternative X is provided in terms of an interval value from the unit interval rather than a precise value, the problem of ordering them must be considered [39].

Thus, to obtain the required ordering in the case of interval-valued information, the golden rule was introduced by Yager [37] to provide a scalar representative value for these intervals.

Definition 3: The golden rule representative value is based on some different rules; this is also called a rule-based description. Basically, all the representative values should be in the unit interval, namely, $\text{Rep}(X) \in [0, 1]$. To better explain its meaning, Yager subsequently implemented the golden rule using the Takagi–Sugeno approach to fuzzy system modeling [41], [56]. To obtain the desired representative value, the following four rules are proposed with the mean and range, which are the core variables of normal intervals [37].

- 1) If the mean of X is large and the range of X is large, then $\text{Rep}(X) = 1$.

- 2) If the mean of X is large and the range of X is small, then $\text{Rep}(X) = 0.5$.
- 3) If the mean of X is small and the range of X is large, then $\text{Rep}(X) = 0.5$.
- 4) If the mean of X is small and the range of X is small, then $\text{Rep}(X) = 0$.

The large and small variables used in the rules above are represented as the linear fuzzy sets L and S defined on the unit interval. Thus, L and S are expressed as

$$L(y) = y, S(y) = 1 - y. \quad (3)$$

Definition 4: Based on the linear fuzzy sets, the TSK approach is used to obtain the representative value associated with an interval value V and its mean m and range r ; namely, for a unique interval $V = [a, b]$, the specific function is shown as [39]

$$\begin{aligned} m &= \frac{a+b}{2} \\ r &= b-a \\ \text{Rep}(V) &= \frac{1m(1-r) + (0.5)mr + (0.5)(1-m)r + 0(1-m)(1-r)}{m(1-r) + mr + (1-m)r + (1-m)(1-r)}. \end{aligned} \quad (4)$$

The result with the simplification operation is

$$\text{Rep}(V) = m + (0.5 - m)r. \quad (5)$$

Then, we can discuss the formula. If $m < 0.5$, the range tends to increase the representative value. In addition, if $m > 0.5$, the range tends to decrease the representative value. While if $m = 0.5$, the representative value is equal to 0.5.

However, $(\partial \text{Rep}(V) / \partial m) = 1 - r \geq 0$. Hence, the representative value always increases as the mean increases; it is monotonic with respect to the mean.

For an interval $V = [a, b]$, the representative value formula can be expressed in terms of a and b as

$$\text{Rep}(V) = \frac{1}{2}(a^2 + 2b - b^2). \quad (6)$$

Specifically, when $m = 0$, the golden rule will be degenerate into the traditional *median* method.

C. Sigmoid Function

The sigmoid function is a well-known function commonly found in biology and is also called the S-type growth curve. In information science, the sigmoid function is often used as a threshold function of a neural network due to its monotonically increasing character and its inverse function. Moreover, the variable is mapped between 0 and 1 [40].

To fully use this kind of function, some researchers have divided the sigmoid function into two parts. The first part is frequently called the log-sigmoid function, and the second part is usually expressed as the tan-sigmoid function. Additionally, both functions are continuous, monotonically increasing, and numerical functions. Here, we simply introduce some of their basic concepts.

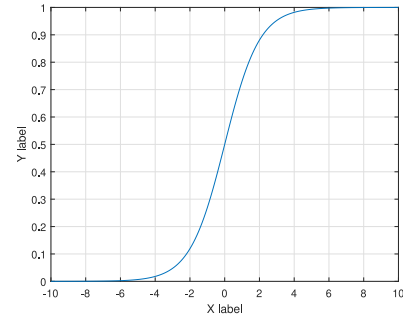


Fig. 1. Schematic of the log-sigmoid function.

1) Log-Sigmoid Function: According to its excellent and special characters, the log-sigmoid function has been applied in many fields, such as machine learning and neural networks [57], [58].

Definition 5: The basic equation of the log-sigmoid function is shown as follows [59]:

$$f(x) = \frac{1}{1 + e^{-x}}. \quad (7)$$

This kind of function has very excellent characteristics in that the domain is the range of real numbers, but the value field is always in the unit interval (0,1). Additionally, the derivative can be calculated by its own value, which is expressed as

$$f'(x) = f(x)(1 - f(x)) = \frac{e^{-x}}{(1 + e^{-x})^2}. \quad (8)$$

The image corresponding to the function expression is shown in Fig. 1.

The log-sigmoid function can be seen everywhere in the field of machine learning. For example, we commonly use it as the response activation function of the output unit in the logistic regression and softmax regression classifiers [60]. The neuron activation function in the neural network is also widely used.

2) Tan-Sigmoid Function: Accordingly, there is also a tan-sigmoid function, which is widely used in backpropagation (BP) neural networks [61], [62].

Definition 6: In a BP neural network, the tan-sigmoid function is often used as a transfer function for the hidden layer and the output layer. The basic expression is shown as follows [63]:

$$\theta(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2}{1 + e^{-2x}} - 1. \quad (9)$$

Similarly, the derivative of the tan-sigmoid function also has a certain correlation with its own value, which is simply expressed as

$$\theta'(x) = 1 - \theta^2(x). \quad (10)$$

The image corresponding to the function expression is shown in Fig. 2.

When constructing the correlation model of the neural network, the above two types of sigmoid functions are often expressed as the conversion functions of the output layer and the hidden layer. They can map the input of the domain in the real set to the interval (0, 1) and (-1, 1). The function

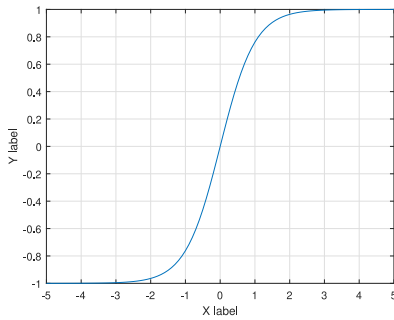


Fig. 2. Schematic of the tan-sigmoid function.

used is usually determined by the relationship between input and output. When the output cannot have a negative value, the log-sigmoid function is used. When the output contains a negative value, it is more appropriate to use the tan-sigmoid function.

III. GENERALIZED GOLDEN RULE

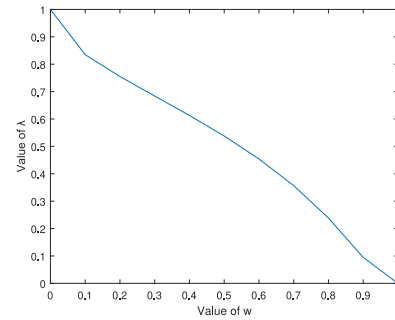
In this section, a generalized golden rule method is proposed to obtain the ordering of different intervals. Based on the basic idea of the golden rule representative value proposed by Yager [37], the sigmoid function is introduced to the process of dynamic comparison to obtain the ordering of different intervals.

Assume that there are several intervals $V_i = [a, b]$ in the comparison system in which their scalar values need to be compared to identify a reasonable order. In addition, all of the scalar values satisfy the condition that $a, b \in [0, 1]$ and $a \leq b$. To explain the problem more intuitively, the notation $V_1 \leq V_2$ is used to express the relationship that V_1 is larger than V_2 .

For an interval, the mean value is widely considered to be the most fundamental variable. Therefore, the core idea is to consider the influence of the range factor on the basis of the mean to more reasonably measure the scalar value of the intervals.

Moreover, when the interval is applied to MCDM that represents the satisfaction of an alternative, the interval often comes with some uncertain information that is incidentally introduced by the decision maker. For example, assuming that alternative x under a condition A is considered to be $C_A(x) = [0.2, 0.4]$, and another alternative y is considered to be $C_A(y) = [0.2, 0.8]$, then it is clear that the uncertainty of the evaluation for alternative x is lower than alternative y ; namely, when evaluating the above two alternatives, the evaluation of alternative x is more convincing, while the result of alternative y is relatively less certain.

Considering the factors mentioned above, the measurement of uncertainty absolutely needs to be introduced for the representative value of an interval. Therefore, under the premise of fully considering the uncertainty of each interval, it is more objective and more reasonable to obtain the representative values of multiple participating objects and to make relevant comparisons, which can also be seen as a kind of *dimensionality reduction* process.

Fig. 3. Schematic of the relationship between λ and w .

In the following, a rule-based description of a representative value for the comparison of intervals is introduced and is subsequently implemented with the TSK fuzzy system model and the sigmoid function.

Assume that n intervals exist that need to be ordered using V_1 to V_n to express. By performing correlation operations on each interval, their respective mean and range can be easily obtained. For instance, assume that we have $V_i = [a_i, b_i]$, then $m_i = [(a_i + b_i)/2]$ and $r_i = b_i - a_i$; namely, there are n intervals in the comparison system.

To consider the uncertain factor of range, we should first focus on the range of all comparison intervals. Thus, a weight determination is proposed to measure the size relationship of all the range values. For interval V_i , the weight is determined as follows:

$$w_i = \frac{r_i}{\sum_{j=1}^n r_j} \quad (11)$$

which satisfies the basic conditions that $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$.

Since the comparison values in the comparison system are different when comparing, the weights assigned are not the same. Moreover, even if the same interval is compared with other different intervals, the weighted results are not always the same. For example, compare an interval A with the remaining intervals B , C , and D and derive a weighted result. Then, compare the interval A with the intervals B and C to obtain a weight value with respect to the interval A . In this situation, unless the interval D is an exact value, that is, its range is 0, the weighting result of the interval A obtained by the two comparison processes is different.

Then, after the weight determination process is completed, a new variable to express the range information is proposed as follows:

$$r'_i = \frac{2}{1 + e^{\sqrt{\frac{w_i}{1-w_i}}}} r_i. \quad (12)$$

The given determination is introduced based on the sigmoid function, which is widely used in BP neural networks. According to the above conditions, the size of w_i is clearly within the range $[0, 1]$. To better show the relationship between the range r_i and the new variable r'_i , the coefficient of r_i can be seen as a variable λ_i with respect to the size of the range weight for the interval V_i . Moreover, to represent the effect of

this parameter on the range value better, the law of change for λ_i is shown in Fig. 3.

As shown in the above figure, the following conclusions can be clearly drawn. Taking the interval V_i as an example, the size of the new variable r_i' satisfies the following laws.

- 1) The greater the r_i , the more comparatively uncertain it is. When r_i is extremely large, it is extremely uncertain. When r_i is equal to 1, it means that it is completely uncertain.
- 2) The smaller the r_i , the more comparatively certain it is. When r_i is extremely small, it is extremely certain. When r_i is equal to 0, it means that it is completely certain.

When the number of members in the comparison system is fixed, if the range of one of the intervals is extended, the corresponding weight w_i also increases, and the value of the variable λ_i decreases, which can be interpreted as the range of the interval is increased to make it more uncertain, and the reduction in the variable λ_i can suppress the effect of the range increase to some extent.

Conversely, when the interval of the interval decreases, the weight occupied in the comparison system decreases, then the variable λ_i increases. It can also be interpreted as the interval of the interval decreases, making it more deterministic, and the increase in the variable λ_i can be expressed to some extent by the effect of increasing the variation in the range.

In the following, the concepts of mean m_i and the new range r_i' associated with an interval value are used to obtain the representation value. In addition, the following four rules are introduced for measuring our desired results.

Rule1: If the mean m_i is large, and the new range r_i' is small, then $\text{Rep}(V) = 1$.

Rule2: If the mean m_i is large, and the new range r_i' is large, then $\text{Rep}(V) = 0.5$.

Rule3: If the mean m_i is small, and the new range r_i' is large, then $\text{Rep}(V) = 0.5$.

Rule4: If the mean m_i is small, and the new range r_i' is small, then $\text{Rep}(V) = 0$.

Here, *large* is represented by linear fuzzy sets L , and *small* is represented by S , $L(x) = x$, $S(x) = 1 - x$, which are all defined on the unit interval. Using the TSK approach to model the rule-based description of the generalized golden rule representative value (GGRV) associated with the interval value V_i , the following equation can be obtained:

$$\begin{aligned} \text{Rep}(V_i) &= \frac{1m_i(1-r_i') + (0.5)m_i r_i' + (0.5)(1-m_i)r_i' + 0(1-m_i)(1-r_i')}{m_i(1-r_i') + m_i r_i' + (1-m_i)r_i' + (1-m_i)(1-r_i')} \end{aligned}$$

Performing the proper algebra, the final equation is expressed as follows:

$$\text{Rep}(V_i) = m_i + (0.5 - m_i)r_i'. \quad (13)$$

According to the above equation, if $m_i < 0.5$, the influence of r_i' is positive and tends to increase the representative value. In addition, if $m > 0.5$, the influence of r_i' is negative and tends to decrease the representative value. Specifically, if $m = 0.5$, the representative value $\text{Rep}(V_i) = 0.5$, which is independent of the uncertainty and the influence of r_i' .

TABLE I
CALCULATION RESULTS IN CASE 1

Interval	m	r	w	λ	r'	Golden rule	Proposed method
[0.1, 0.4]	0.2500	0.3000	0.3333	0.6605	0.1981	0.3250	0.2995
[0.2, 0.5]	0.3500	0.3000	0.3333	0.6605	0.1981	0.3950	0.3797
[0.3, 0.6]	0.4500	0.3000	0.3333	0.6605	0.1981	0.4650	0.4599

TABLE II
CALCULATION RESULTS IN CASE 2

Interval	m	r	w	λ	r'	Golden rule	Proposed method
[0.4, 0.7]	0.5500	0.3000	0.2500	0.7191	0.2157	0.5350	0.5392
[0.5, 0.8]	0.6500	0.3000	0.2500	0.7191	0.2157	0.6050	0.6176
[0.6, 0.9]	0.7500	0.3000	0.2500	0.7191	0.2157	0.6750	0.6961
[0.7, 1.0]	0.8500	0.3000	0.2500	0.7191	0.2157	0.7450	0.7745

Moreover, the representative value can be proven monotonic with respect to the mean m_i

$$V_i' = \frac{\partial \text{Rep}(V_i)}{\partial m} = 1 - r_i' \geq 0.$$

Thus, the representative value always increases as the mean m_i increases. Based on the representative value equation, to find the boundary condition and some properties.

- 1) When $r_i = 0$, regardless of what other intervals in the system are compared, r_i' is equal to 0. In addition, the variable λ_i is the maximum value 1, and the generalized golden rule degenerates to the *midpoint* method.
- 2) When an interval in which the range is nonzero is compared with another interval in which the range is 0, the representative value degenerates to the *midpoint* method.

The flowchart of the calculation process is shown in Fig. 4.

IV. NUMERICAL EXAMPLES

In this section, some specific numerical examples of the calculation with some explanations are illustrated to show the whole process of our proposed method.

Example 1:

Case 1: Assume that there are three intervals, $V_1 = [a_1, b_1]$, $V_2 = [a_2, b_2]$, and $V_3 = [a_3, b_3]$, where $V_1 = [0.1, 0.4]$, $V_2 = [0.2, 0.5]$, and $V_3 = [0.3, 0.6]$. After calculation, the following results can be obtained in Table I.

Case 2: For $V_4 = [0.4, 0.7]$, $V_5 = [0.5, 0.8]$, $V_6 = [0.6, 0.9]$, and $V_7 = [0.7, 1.0]$, the following results can be obtained in Table II.

Obviously, when the original range between the comparison regions is equal but the respective mean is different, the obtained GGRV is not much different from the *golden rule* method, and the size order relationship can be correctly reflected.

Moreover, by comparing the results in cases 1 and 2, it is shown that when the interval ranges are the same, the larger mean indicates that the greater the overall support, the smaller the difference between the results of the proposed method and the results of the *golden rule* method. The results also showed that the difference is greater than the results of the *golden rule* method.

Example 2: Assume that there are five intervals, $V_1 = [a_1, b_1]$, $V_2 = [a_2, b_2]$, $V_3 = [a_3, b_3]$, $V_4 = [a_4, b_4]$, and $V_5 = [a_5, b_5]$, where $V_1 = [0.1, 0.1]$, $V_2 = [0.1, 0.3]$, $V_3 = [0.1, 0.5]$, $V_4 = [0.1, 0.7]$, and $V_5 = [0.1, 0.9]$. After the

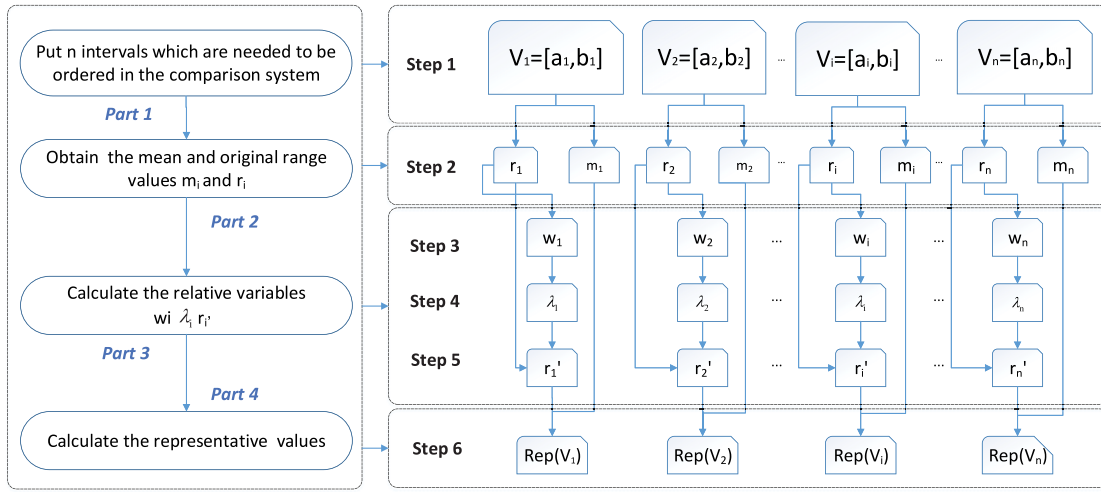


Fig. 4. Flowchart of the calculation process.

TABLE III
CALCULATION RESULTS IN EXAMPLE 2

Interval	m	r	w	λ	r'	Golden rule	Proposed method
[0.1, 0.1]	0.1000	0.0000	0.0000	1.0000	0.0000	0.1000	0.1000
[0.1, 0.3]	0.2000	0.2000	0.1000	0.8349	0.1670	0.2600	0.2501
[0.1, 0.5]	0.3000	0.4000	0.2000	0.7551	0.3020	0.3800	0.3604
[0.1, 0.7]	0.4000	0.6000	0.3000	0.6839	0.4103	0.4600	0.4410
[0.1, 0.9]	0.5000	0.8000	0.4000	0.6130	0.4904	0.5000	0.5000

TABLE IV
CALCULATION RESULTS IN EXAMPLE 3

Interval	m	r	w	λ	r'	Golden rule	Proposed method
[0.5, 0.5]	0.5000	0.0000	0.0000	1.0000	0.0000	0.5000	0.5000
[0.4, 0.6]	0.5000	0.2000	0.1000	0.8349	0.1670	0.5000	0.5000
[0.3, 0.7]	0.5000	0.4000	0.2000	0.7551	0.3020	0.5000	0.5000
[0.2, 0.8]	0.5000	0.6000	0.3000	0.6839	0.4103	0.5000	0.5000
[0.1, 0.9]	0.5000	0.8000	0.4000	0.6130	0.4904	0.5000	0.5000

process of calculation, the following results can be obtained in Table III.

Clearly, the values of the left endpoints of the five intervals in the comparison system are the same in Example 2, but the ranges are different. The calculated results are shown above, which shows that the representative value increases as the interval mean increases. Specifically, when the mean of the interval is 0.5 or the interval range is 0, the proposed method degenerates to the *midpoint* method, which is the same as the *golden rule* method.

Example 3: Assume that there are five intervals, $V_1 = [a_1, b_1]$, $V_2 = [a_2, b_2]$, $V_3 = [a_3, b_3]$, $V_4 = [a_4, b_4]$, and $V_5 = [a_5, b_5]$, where $V_1 = [0.5, 0.5]$, $V_2 = [0.4, 0.6]$, $V_3 = [0.3, 0.7]$, $V_4 = [0.2, 0.8]$, and $V_5 = [0.1, 0.9]$. After calculation, the following results can be obtained in Table IV.

In the five intervals shown in Example 3, the values of the mean are all 0.5, but each of them has a different range.

When the range is 0.5, both the *golden rule* method and the proposed method degenerate to the *midpoint* method, which shows the calculation result of the representative value and the independence of the uncertainty information given by the range when $m = 0.5$.

Example 4: Assume that there are five intervals, $V_1 = [a_1, b_1]$, $V_2 = [a_2, b_2]$, $V_3 = [a_3, b_3]$, $V_4 = [a_4, b_4]$, and

TABLE V
CALCULATION RESULTS IN EXAMPLE 4

Interval	m	r	w	λ	r'	Golden Rule	Proposed method
[0.0, 0.9]	0.4500	0.9000	0.3600	0.6416	0.5775	0.4950	0.4789
[0.1, 0.8]	0.4500	0.7000	0.2800	0.6979	0.4885	0.4850	0.4744
[0.2, 0.7]	0.4500	0.5000	0.2000	0.7551	0.3775	0.4750	0.4689
[0.3, 0.6]	0.4500	0.3000	0.1200	0.8174	0.2452	0.4650	0.4623
[0.4, 0.5]	0.4500	0.1000	0.0400	0.8983	0.0898	0.4550	0.4545

$V_5 = [a_5, b_5]$, where $V_1 = [0.0, 0.9]$, $V_2 = [0.1, 0.8]$, $V_3 = [0.2, 0.7]$, $V_4 = [0.3, 0.6]$, and $V_5 = [0.4, 0.5]$. After calculation, the following results can be obtained in Table V.

As shown in Example 4, although the median points of the five intervals are the same, they are not 0.5. Evidently, when the range is larger, the proposed method shows little difference from the golden rule. In addition, as the range is larger, the difference is also larger, and the smaller the range, the smaller the difference.

V. APPLICATION

In this section, some applications are analyzed to illustrate the practicality of the proposed method.

A. Background Concepts of Interval-Valued Criteria Satisfaction

For the following discussion, there exists some uncertainty in the determination of the satisfaction for a criteria C_i by an alternative x [39], [64]. Generally, the assessment provided by experts often meets certain uncertainty information. From a realistic point of view, the evaluation results expressed as intervals are more objective than the traditional single-value evaluation results. In some practical applications, interval satisfaction can usually avoid information loss due to excessive accuracy. Hence, the usage of interval satisfaction is quite essential in the real world.

To illustrate the problem better, the framework of the MCDM problems with satisfaction intervals is shown in Fig. 5.

B. Application 1

Assume that each $C_i(x)$ is expressed as an interval value, which denotes that $C_i(x) = [a_i(x), b_i(x)]$. The core work

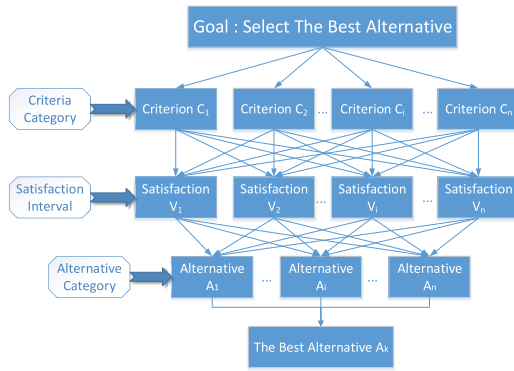


Fig. 5. Framework of the MCDM problems with satisfaction intervals.

TABLE VI
CALCULATION RESULTS OF THE RELATIVE VARIABLES OF $C_i(x)$

Interval	Index	w	λ	r'	Proposed method
[0.2, 0.7]	1	0.5556	0.4928	0.2464	0.4623
[0.4, 0.8]	2	0.4444	0.5804	0.2322	0.5768
[0.5, 0.5]	3	0.0000	1.0000	0.0000	0.5000

is to evaluate $D(x) = \text{Max}[T(C_{\lambda(j)}(x), \mu(H_j))]$, where $j = 1$ to q . Moreover, if $C_{\lambda(j)}(x) = [a_{\lambda(j)}(x), b_{\lambda(j)}(x)]$, then $T(C_{\lambda(j)}(x), \mu(H_j))$ can be calculated as follows [39]:

$$T(C_{\lambda(j)}(x), \mu(H_j)) = [T(a_{\lambda(j)}(x), \mu(H_j)), T(b_{\lambda(j)}(x), \mu(H_j))].$$

Suppose that we have three criteria, C_1 , C_2 , and C_3 , and their respective satisfactions by an alternative x are interval valued and are denoted by $C_1(x)$, $C_2(x)$, and $C_3(x)$. Here, the specific values are given as follows:

$$C_1(x) = [0.2, 0.7], C_2(x) = [0.4, 0.8], C_3(x) = [0.5, 0.5].$$

Step 1: Obtain their mean values and respective original range values. The results are shown as follows:

$$\begin{aligned}
 m_1(x) &= \frac{0.2 + 0.7}{2} = 0.45 \\
 m_2(x) &= \frac{0.4 + 0.8}{2} = 0.60 \\
 m_3(x) &= \frac{0.5 + 0.5}{2} = 0.50 \\
 r_1(x) &= 0.7 - 0.2 = 0.5 \\
 r_2(x) &= 0.8 - 0.4 = 0.4 \\
 r_3(x) &= 0.5 - 0.5 = 0.0.
 \end{aligned}$$

Step 2: Calculate their representative values using the proposed method and obtain the order by comparing the calculated results for their representative values shown in Table VI.

Thus, we have $\lambda(1) = 2$, $\lambda(2) = 3$, and $\lambda(3) = 1$, which means that $C_2(x) > C_3(x) > C_1(x)$.

Step 3: Obtain the result of $\mu(H_j)$. According to the above calculation results, (H_j) can be expressed as

$$\begin{aligned}
 H_1 &= \{C_2\} \\
 H_2 &= \{C_2, C_3\} \\
 H_3 &= \{C_2, C_3, C_1\}.
 \end{aligned}$$

TABLE VII
CALCULATION RESULTS OF THE RELATIVE VARIABLES OF $T(C_{\lambda(j)}(x), \mu(H_j))$

Interval	Index	m	r	w	λ_i	r'	Representative value
[0.4, 0.4]	1	0.4000	0.0000	0.0000	1.0000	0.0000	0.4000
[0.5, 0.5]	2	0.5000	0.0000	0.0000	1.0000	0.0000	0.5000
[0.2, 0.7]	3	0.4500	0.5000	1.0000	0.0000	0.0000	0.4500

TABLE VIII
SIS OF TEN SUPPLIERS

Supplier	SIS	Supplier	SIS
S_1	[0.2, 0.7]	S_6	[0.4, 0.5]
S_2	[0.5, 0.6]	S_7	[0.5, 0.7]
S_3	[0.8, 0.9]	S_8	[0.2, 0.5]
S_4	[0.0, 1.0]	S_9	[0.2, 0.6]
S_5	[0.3, 0.3]	S_{10}	[0.3, 0.6]

TABLE IX
CALCULATION RESULTS IN EXAMPLE 5

Interval	m	r	w	λ	r'	Proposed method	Rank
[0.2, 0.7]	0.4500	0.5000	0.1667	0.7800	0.3900	0.4695	5
[0.5, 0.6]	0.5500	0.1000	0.0333	0.9074	0.0907	0.5455	3
[0.8, 0.9]	0.8500	0.1000	0.0333	0.9074	0.0907	0.8182	1
[0.0, 1.0]	0.5000	1.0000	0.3333	0.6605	0.6605	0.5000	4
[0.3, 0.3]	0.3000	0.0000	0.0000	1.0000	0.0000	0.3000	10
[0.4, 0.5]	0.4500	0.1000	0.0333	0.9074	0.0907	0.4545	7
[0.5, 0.7]	0.6000	0.2000	0.0667	0.8672	0.1734	0.5827	2
[0.2, 0.5]	0.3500	0.3000	0.1000	0.8349	0.2505	0.3876	9
[0.2, 0.6]	0.4000	0.4000	0.1333	0.8064	0.3225	0.4323	8
[0.3, 0.6]	0.4500	0.3000	0.1000	0.8349	0.2505	0.4625	6

TABLE X
NORMALIZED WEIGHTS FOR ALL THE CRITERIA

Criteria	C_1	C_2	C_3	C_4	C_5
ω	0.30298	0.25329	0.17542	0.16883	0.09948

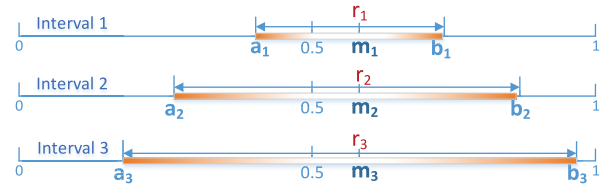


Fig. 6. Different intervals with the same mean value.

Additionally, we first assume that there exists the following fuzzy measure μ with respect to the given criteria [65], [66]:

$$\begin{aligned}
 \mu(\{C_1\}) &= 0.3, \mu(\{C_2\}) = 0.4, \mu(\{C_3\}) = 0.3 \\
 \mu(\{C_1, C_2\}) &= 0.8, \mu(\{C_1, C_3\}) = 0.5, \mu(\{C_2, C_3\}) = 0.9 \\
 \mu(\{C_1, C_2, C_3\}) &= 1.
 \end{aligned}$$

Thus, the following equation can be easily obtained with the given fuzzy measure:

$$\begin{aligned}
 \mu(H_1) &= \mu(\{C_2\}) = 0.4 \\
 \mu(H_2) &= \mu(\{C_2, C_3\}) = 0.9 \\
 \mu(H_3) &= \mu(\{C_2, C_3, C_1\}) = 1.
 \end{aligned}$$

Step 4: Calculate the value of $T(C_{\lambda(j)}(x), \mu(H_j))$. In this situation, we simply assume that T is the min norm.

TABLE XI
AGGREGATED NORMALIZED INTERVAL-VALUED HESITANT FUZZY DECISION MATRIX

A_i	C_1	C_2	C_3	C_4	C_5
A_1	[0.2348, 0.3351]	[0.7379, 0.8413]	[0.2000, 0.3000]	[0.6366, 0.7379]	[0.5358, 0.6366]
A_2	[0.2681, 0.3684]	[0.6698, 0.7711]	[0.2681, 0.3684]	[0.7116, 0.8183]	[0.6698, 0.7711]
A_3	[0.1680, 0.2681]	[0.6698, 0.7711]	[0.1347, 0.2348]	[0.7379, 0.8413]	[0.5783, 0.6825]

To illustrate clearly, $B_j(x)$ is introduced, which satisfies the equation that $B_j(x) = T(C_{\lambda(j)}(x), \mu(H_j))$. Thus, we can obtain that

$$B_j(x) = [a_{\lambda(j)}(x) \wedge \mu(H_j), b_{\lambda(j)}(x) \wedge \mu(H_j)].$$

And

$$B_1(x) = [0.4 \wedge 0.4, 0.8 \wedge 0.4] = [0.4, 0.4]$$

$$B_2(x) = [0.5 \wedge 0.9, 0.5 \wedge 0.9] = [0.5, 0.5]$$

$$B_3(x) = [0.2 \wedge 1.0, 0.7 \wedge 1.0] = [0.2, 0.7].$$

Step 5: Calculate the representative values using the proposed method to order the value of $T(C_{\lambda(j)}(x), \mu(H_j))$ to obtain $D(x)$.

According to the proposed method, the calculation results can be shown in Table VII.

Thus, we can obtain the order that $B_1(x) < B_3(x) < B_2(x)$, and $D(x) = \text{Max}[B_1(x), B_2(x), B_3(x)] = B_2(x) = [0.5, 0.5]$ in this situation.

To illustrate the application process, the above process simply calculates an alternative x . However, in more complicated cases, there will be more possible alternatives $X = [x_1, x_2, \dots, x_n]$ for comparison. In that case, the maximum of $D(x)$ must be selected to obtain the best alternative.

C. Application 2

Manufacturer plans to purchase a batch of new equipment from the market. There are ten suppliers $\{S_1, S_2, \dots, S_{10}\}$ are taken into consideration. Several experts consider them from four aspects, such as environment production, quality of equipment, production efficiency, and economy. Note that all the satisfaction provided by experts are considered as the form of interval. After the aggregation process, the synthesized interval satisfaction (SIS) are achieved, which are denoted as $V = \{V_1, V_2, \dots, V_{10}\}$ and the details are shown in Table VIII.

After the calculation process, the following findings can be achieved, which are shown in Table IX. We can see that supplier S_3 could be preferentially considered as the best selection and supplier S_5 is possibly regarded as the worst one.

VI. COMPARISON AND DISCUSSION

Traditionally, to measure the size of the interval, the *midpoint* method is used. However, using this method causes some misunderstandings in some special situations.

As shown in Fig. 6, the usage of the simple surrogate for the interval, in certain cases, may cause the situation that no fully feedbacks to the preferences for some decision makers. For example, we suppose the outcomes of an alternative x_i with criteria C_i are given by three experts as shown in Fig. 6. If decision makers use the traditional midpoint method to compare the results, they will be notified of the outcomes without

TABLE XII
CALCULATION RESULTS FOR ALL THE GREEN SUPPLIERS

Green supplier	Interval	GGRV	Rank
A_1	[0.4536, 0.5548]	0.5039	2
A_2	[0.4844, 0.5861]	0.5329	1
A_3	[0.4260, 0.5274]	0.4783	3

completely consistent evaluation. Furthermore, it may cause serious information loss and lead to the unpredictable cost in some complicated application environments, if we only keep using the traditional midpoint method without the functions to measure uncertain information.

Therefore, Yager [37] proposed a rule-based approach to solving this problem named the *golden rule* representative value method. Although the *golden rule* method has many great advantages, the impact of the influence of other interval ranges is not considered in the comparison system with uncertain information. To address this issue, the proposed method collects the range information of all intervals in comparison system and then introduce the sigmoid function in BP neural networks to tune parameters. Compared with traditional approaches, the representative values of the individual interval are dynamically adjusted with the change of other intervals. In other words, for any intervals, the uncertain information is not only related to its range but also related to other intervals in comparison system, which can be seen as the *relative uncertainty*. Furthermore, the proposed method retains the basic idea of the *golden rule* with the usage of the TSK fuzzy model.

As shown in Section IV, the calculation results in many different cases are listed. Thus, it is simple to find the following.

- 1) The proposed method is completely data driven, and it extracts the implicit uncertain information based on the interval data. Thus, it considers the influence of the interval range on its representative value.
- 2) Taking into account the influence of the uncertainty of each interval in the comparison system, the proposed method embodies a process of dynamic comparison; that is, the representative value used to represent the interval will change when the interval for comparison is different.
- 3) Introducing the sigmoid function into the BP neural network can clearly indicate the uncertainty relationship reflected in the reasonable constraint effect of the parameter λ on the range and well limit the range of the parameter.
- 4) Retaining the basic idea of the *golden rule* proposed by Yager and using the TSK fuzzy model to make the proposed method are consistent with the advantages.

TABLE XIII
SENSITIVE ANALYSIS CALCULATION RESULTS IN SECTION VII

Case 1				Case 2				Case 3			
ϵ values	Interval	GGRV	Rank	ϵ values	Interval	GGRV	Rank	ϵ values	Interval	GGRV	Rank
$\epsilon = 0.00$	[0.20 , 0.60]	0.4245	1	$\epsilon = 0.00$	[0.30 , 0.70]	0.5000	1	$\epsilon = 0.00$	[0.40 , 0.80]	0.5755	5
	[0.25 , 0.55]	0.4205	2		[0.35 , 0.65]	0.5000	1		[0.45 , 0.75]	0.5795	4
	[0.30 , 0.50]	0.4151	3		[0.40 , 0.60]	0.5000	1		[0.50 , 0.70]	0.5849	3
	[0.35 , 0.45]	0.4083	4		[0.45 , 0.55]	0.5000	1		[0.55 , 0.65]	0.5917	2
	[0.40 , 0.40]	0.4000	5		[0.50 , 0.50]	0.5000	1		[0.60 , 0.60]	0.6000	1
$\epsilon = 0.05 * r_1$	[0.20 - ϵ , 0.60 + ϵ]	0.4262	1	$\epsilon = 0.05 * r_1$	[0.30 - ϵ , 0.70 + ϵ]	0.5000	1	$\epsilon = 0.05 * r_1$	[0.40 - ϵ , 0.80 + ϵ]	0.5738	5
	[0.25 , 0.55]	0.4207	2		[0.35 , 0.65]	0.5000	1		[0.45 , 0.75]	0.5792	4
	[0.30 , 0.50]	0.4152	3		[0.40 , 0.60]	0.5000	1		[0.50 , 0.70]	0.5848	3
	[0.35 , 0.45]	0.4084	4		[0.45 , 0.55]	0.5000	1		[0.55 , 0.65]	0.5916	2
	[0.40 , 0.40]	0.4000	5		[0.50 , 0.50]	0.5000	1		[0.60 , 0.60]	0.6000	1
$\epsilon = 0.05 * r_2$	[0.20 , 0.60]	0.4249	1	$\epsilon = 0.05 * r_2$	[0.30 , 0.70]	0.5000	1	$\epsilon = 0.05 * r_2$	[0.40 , 0.80]	0.5751	5
	[0.25 - ϵ , 0.55 + ϵ]	0.4221	2		[0.35 - ϵ , 0.65 + ϵ]	0.5000	1		[0.45 - ϵ , 0.75 + ϵ]	0.5779	4
	[0.30 , 0.50]	0.4152	3		[0.40 , 0.60]	0.5000	1		[0.50 , 0.70]	0.5848	3
	[0.35 , 0.45]	0.4084	4		[0.45 , 0.55]	0.5000	1		[0.55 , 0.65]	0.5916	2
	[0.40 , 0.40]	0.4000	5		[0.50 , 0.50]	0.5000	1		[0.60 , 0.60]	0.6000	1
$\epsilon = 0.05 * r_3$	[0.20 , 0.60]	0.4247	1	$\epsilon = 0.05 * r_3$	[0.30 , 0.70]	0.5000	1	$\epsilon = 0.05 * r_3$	[0.40 , 0.80]	0.5753	5
	[0.25 , 0.55]	0.4206	2		[0.35 , 0.65]	0.5000	1		[0.45 , 0.75]	0.5794	4
	[0.30 - ϵ , 0.50 + ϵ]	0.4164	3		[0.40 - ϵ , 0.60 + ϵ]	0.5000	1		[0.50 - ϵ , 0.70 + ϵ]	0.5836	3
	[0.35 , 0.45]	0.4084	4		[0.45 , 0.55]	0.5000	1		[0.55 , 0.65]	0.5916	2
	[0.40 , 0.40]	0.4000	5		[0.50 , 0.50]	0.5000	1		[0.60 , 0.60]	0.6000	1
$\epsilon = 0.05 * r_4$	[0.20 , 0.60]	0.4246	1	$\epsilon = 0.05 * r_4$	[0.30 , 0.70]	0.5000	1	$\epsilon = 0.05 * r_4$	[0.40 , 0.80]	0.5754	5
	[0.25 , 0.55]	0.4206	2		[0.35 , 0.65]	0.5000	1		[0.45 , 0.75]	0.5794	4
	[0.30 , 0.50]	0.4151	3		[0.40 , 0.60]	0.5000	1		[0.50 , 0.70]	0.5849	3
	[0.35 - ϵ , 0.45 + ϵ]	0.4091	4		[0.45 - ϵ , 0.55 + ϵ]	0.5000	1		[0.55 - ϵ , 0.65 + ϵ]	0.5909	2
	[0.40 , 0.40]	0.4000	5		[0.50 , 0.50]	0.5000	1		[0.60 , 0.60]	0.6000	1
$\epsilon = 0.05 * r_5$	[0.20 , 0.60]	0.4245	1	$\epsilon = 0.05 * r_5$	[0.30 , 0.70]	0.5000	1	$\epsilon = 0.05 * r_5$	[0.40 , 0.80]	0.5755	5
	[0.25 , 0.55]	0.4205	2		[0.35 , 0.65]	0.5000	1		[0.45 , 0.75]	0.5795	4
	[0.30 , 0.50]	0.4151	3		[0.40 , 0.60]	0.5000	1		[0.50 , 0.70]	0.5849	3
	[0.35 , 0.45]	0.4083	4		[0.45 , 0.55]	0.5000	1		[0.55 , 0.65]	0.5917	2
	[0.40 - ϵ , 0.40 + ϵ]	0.4000	5		[0.50 - ϵ , 0.50 + ϵ]	0.5000	1		[0.60 - ϵ , 0.60 + ϵ]	0.6000	1

Furthermore, our proposed method would also be suitable for the real-life scenarios, by comparing with existing methods in the interval-valued environment. For instance, Gitinavard *et al.* [67] introduced a novel interval-valued hesitant fuzzy group outranking approach, which has been proved the feature of practicality for proceeding with the green supplier selection. Here, we utilized the example shown in [67] to carry out the comparative analysis process. In specific, three green suppliers (A_1 , A_2 , and A_3) are required to be considered with five criteria which are shown as follows: 1) C_1 : cost; 2) C_2 : quality; 3) C_3 : delivery; 4) C_4 : technology capability; and 5) C_5 : environmental competency.

The normalized weights ω for all the criteria and the aggregated normalized interval-valued hesitant fuzzy decision matrix are shown in Tables X and XI, respectively.

Completed weighted averaging aggregation process, the ranking outcomes for all the green suppliers based on the proposed method are shown in Table XII.

Therefore, the second and third candidate green suppliers are considered as the best and worst selection, respectively, which is consistent with the results achieved in [67].

In Section V, the result of the MCDM problem with interval-valued criteria satisfactions is discussed by comparing the parts, in which the order is needed in the decision process. In addition, the rationality of the proposed method can also be observed.

VII. SENSITIVITY ANALYSIS

In this section, we include a sensitivity analysis to indicate the robustness and sensitiveness of our proposed approach. At the initial step, we determine the parameter ϵ to express the

possible error caused by experts. Then, with the data-driven principle, we assume that there are five intervals in comparison system where four of them are fixed. The parameter ϵ is constrained to be less than 0.05, which can be regarded as a proper deviation for error factor.

The five intervals in case 1 are determined as $V_1 = [0.20, 0.60]$, $V_2 = [0.25, 0.55]$, $V_3 = [0.30, 0.50]$, $V_4 = [0.35, 0.45]$, and $V_5 = [0.40, 0.40]$. We note that the midpoints of all of them are 0.4. In terms of case 2, all of the intervals' midpoints are fixed as 0.5. Then, we note all the midpoints of intervals shown in case 3 are 0.6. In this case, if we simply use the *midpoint* method to measure its size, the effective ranking outcomes cannot be achieved. Due to the different error level reflected by different ϵ values, the calculation results of its GGRVs are shown as follows.

As shown in Table XIII, we note the ranking outcomes cannot be changed with the introduction of the error factor, suggesting the robustness, and sensitiveness of our proposed approach. Specifically, considering that the range of unit interval is 1, the range values of intervals within the unit interval are usually less than 1. Thus, the difference between the left and right endpoints of our selected five intervals is constantly set to 0.05, which indicates that the error factor cannot take an influence on the uncertainty relation of such intervals in comparison system. Furthermore, the ranking outcomes reflecting the features of the *golden rule* are shown in Section III.

VIII. CONCLUSION

In this paper, we proposed a novel approach to order the intervals based on the *golden rule*. This approach introduces

the modified formulas, definitions, and parameters, which considers the uncertainty information created by the range value and intervals' effects in an information system. The goal of this approach is to improve the MCDM problems when the satisfaction for all the alternatives are interval valued. As the traditional *midpoint* method cannot develop a model to deal with the uncertainty information, our proposed approach is a complementary model to improve the uncertainties related problems. Our results show the greater the uncertainty and the smaller their representative values for the intervals in a comparison system. This finding is consistent with the realistic MCDM scenario.

It should be noted that there are still some limitations of the proposed method. For instance, to illustrate the problem, the proposed method has not been applied in solving more complex decision situations. Moreover, we believe that it can be used to solve other kinds of problems. This issue is one of our ongoing works.

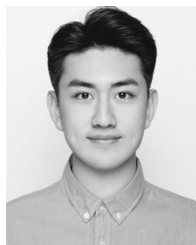
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